FOURIER-MUKAI TRANSFORMS FOR QUOTIENT VARIETIES

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ABSTRACT. We study Fourier-Mukai transforms for smooth projective varieties whose canonical bundles have finite order. Our results lead to new transforms for Enriques and bielliptic surfaces.

1. Introduction

Fourier-Mukai transforms are now well-established as a useful tool for computing moduli spaces of sheaves on smooth projective varieties [3], [9]. More recently there has been further interest in these transforms because of their connection with homological mirror symmetry [8].

In this paper we study Fourier-Mukai transforms for smooth complex projective varieties whose canonical bundles have finite order, and relate them to equivariant transforms on certain finite covering spaces. Applying our results to the case of Enriques and bielliptic surfaces, we obtain new examples of transforms for complex surfaces. These results will be used in [5], where we find all pairs of minimal complex surfaces with equivalent derived categories.

A Fourier-Mukai (FM) transform is an exact equivalence

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

between the bounded derived categories of coherent sheaves on two smooth projective varieties X and Y. Due to a result of D. Orlov [14], it is known that for any such equivalence there is an object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$ and an isomorphism of functors

$$\Phi(-) \cong \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_Y^*(-)),$$

where $Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$ are the projection maps.

Let X be a smooth complex projective variety, and suppose that ω_X has finite order n say. Then there is a finite unbranched cover of X by a smooth projective variety $\widetilde{X}$ with trivial canonical bundle. Moreover,

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X is the quotient of $\widetilde{X}$ by an action of the cyclic group of order n. We call the quotient morphism

$$p_X : \widetilde{X} \to X$$

the *canonical cover* of X.

Suppose Y is another smooth projective variety, and

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

is a FM transform. We show that ω_Y also has order n, and that if $p_Y : \widetilde{Y} \to Y$ is the canonical cover of Y, there is a $\mathbb{Z}_n$-equivariant FM transform

$$\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$$

such that the following two squares of functors commute

$$\begin{array}{ccc} \mathrm{D}(\widetilde{Y}) & \xrightarrow{\widetilde{\Phi}} & \mathrm{D}(\widetilde{X}) \\ {\scriptstyle p_Y^*}\uparrow\downarrow{\scriptstyle p_{Y,*}} & & {\scriptstyle p_X^*}\uparrow\downarrow{\scriptstyle p_{X,*}} \\ \mathrm{D}(Y) & \xrightarrow{\Phi} & \mathrm{D}(X). \end{array}$$

Conversely, if $\widetilde{Y}$ is a smooth projective variety with trivial canonical bundle, and a $\mathbb{Z}_n$-action with smooth quotient Y, and

$$\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$$

is a $\mathbb{Z}_n$-equivariant FM transform, then the quotient map $p_Y : \widetilde{Y} \to Y$ is a canonical cover, and there is a FM transform Φ such that the diagram above commutes.

Notation. All varieties will be over the complex number field $\mathbb{C}$. Given a projective variety X, the category of coherent $\mathcal{O}_X$-modules will be denoted $\mathrm{Mod}(X)$. The bounded derived category of coherent sheaves on X is denoted $\mathrm{D}(X)$. Its objects are bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology sheaves. We refer to [6] for details on derived categories. Note that, as is usual, the translation functor on $\mathrm{D}(X)$ is written [1], so that the symbol $E[m]$ means the object E of $\mathrm{D}(X)$ shifted to the left by m places.

By a sheaf on X we mean an object of $\mathrm{Mod}(X)$, and a point of X always means a closed (or geometric) point. The structure sheaf of such a point $x \in X$ will be denoted $\mathcal{O}_x$. The canonical sheaf of a smooth projective variety X is denoted ω_X.

2. Canonical covers

2.1. If X is a smooth projective variety whose canonical bundle has finite order, one expects a degree n cover of X corresponding to the element $[\mathrm{c}_1(\omega_X)] \in \pi_1(X)$. This is the canonical cover of X referred to in the introduction.

Proposition 2.1. *Let X be a smooth projective variety whose canonical bundle has finite order n. Then there is a smooth projective variety $\widetilde{X}$ with trivial canonical bundle, and an étale cover $p : \widetilde{X} \to X$ of degree n, such that*

$$p_* \mathcal{O}_{\widetilde{X}} \cong \bigoplus_{i=0}^{n-1} \omega_X^i. \tag{1}$$

Furthermore, $\widetilde{X}$ is uniquely defined up to isomorphism, and there is a free action of the cyclic group $G = \mathbb{Z}_n$ on $\widetilde{X}$ such that $p : \widetilde{X} \to X = \widetilde{X}/G$ is the quotient morphism.

Proof. By the results of [1], §I.17, there exists a smooth projective variety $\widetilde{X}$ and a degree n unbranched cover satisfying (1). Furthermore $\omega_{\widetilde{X}} = p^* \omega_X = \mathcal{O}_{\widetilde{X}}$. By [7], Ex. II.5.17, $\widetilde{X}$ is isomorphic to $\mathbf{Spec}(\mathcal{A})$, where $\mathcal{A}$ is the $\mathcal{O}_X$-algebra

$$\bigoplus_{i=0}^{n-1} \omega_X^i,$$

which proves uniqueness. The action of G is generated by the automorphism $\otimes \omega_X$ of $\mathcal{A}$, and clearly $X = \widetilde{X}/G$. $\square$

Definition 2.2. Let X be a smooth projective variety whose canonical bundle has finite order n. By the *canonical cover* of X we shall mean the unique smooth projective variety $\widetilde{X}$ of Proposition 2.1, together with the quotient morphism $p_X : \widetilde{X} \to X$.

Examples 2.3. (a) An Enriques surface is a smooth surface X with $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ whose canonical bundle has order 2. The canonical cover of such a surface is a K3 surface $\widetilde{X}$, and X is the quotient of $\widetilde{X}$ by the group generated by a fixed-point-free automorphism of order 2. See [1], Ch. VIII.

(b) A bielliptic surface is a smooth surface X with $\mathrm{H}^1(X, \mathcal{O}_X) = \mathbb{C}^2$ whose canonical bundle has finite order $n > 1$. The possible values of n are $2, 3, 4$ and 6. The canonical cover of such a surface is an Abelian surface $\widetilde{X}$, and X is the quotient of $\widetilde{X}$ by a free action of a cyclic group of automorphisms of order n. See [1], §V.5.

2.2. Let X be a smooth projective variety whose canonical bundle has finite order n, and let

$$p : \widetilde{X} \to X$$

be the canonical cover. Thus X is the quotient of $\widetilde{X}$ by a free action of $G = \mathbb{Z}_n$. Let g be a generator of G.

Let G-Mod$(\widetilde{X})$ be the category of G-equivariant sheaves on $\widetilde{X}$. Since G is cyclic, a sheaf E on $\widetilde{X}$ is equivariant if and only if $g^*(E) \cong E$.

Similarly, we let Sp-Mod(X) denote the category of coherent $p_*(\mathcal{O}_{\widetilde{X}})$-modules on X. A sheaf E on X is in this category if and only if $E \otimes \omega_X \cong E$. Following [13] we call these sheaves *special*.

Lemma 2.4. *The functors*

$$p^* : \mathrm{Mod}(X) \longrightarrow \text{G-Mod}(\widetilde{X}),$$

and

$$p_* : \mathrm{Mod}(\widetilde{X}) \longrightarrow \text{Sp-Mod}(X),$$

are equivalences of categories.

Proof. This is standard. For the first part see [12], §7. The second part follows from [7], Ex. II.5.17 (e). See also [13], Prop. 1.2. □

We need to generalise this result to include complexes of sheaves.

Proposition 2.5. *(a) Let $\widetilde{E}$ be an object of $\mathrm{D}(\widetilde{X})$. Then there is an object E of $\mathrm{D}(X)$ such that $p^*E \cong \widetilde{E}$ if and only if there is an isomorphism $g^*\widetilde{E} \cong \widetilde{E}$.*

(b) Let E be an object of $\mathrm{D}(X)$. Then there is an object $\widetilde{E}$ of $\mathrm{D}(\widetilde{X})$ such that $p_*\widetilde{E} \cong E$ if and only if there is an isomorphism $E \otimes \omega_X \cong E$.

Proof. We shall prove (a); (b) is entirely analogous. One implication is easy, so let us assume that there is an isomorphism

$$s : \widetilde{E} \longrightarrow g^*\widetilde{E},$$

and find an object E of $\mathrm{D}(X)$ such that

$$p^*E \cong \widetilde{E}.$$

We use induction on the number r of non-zero cohomology sheaves of $\widetilde{E}$. Shifting E if neccesary, let us assume that $H^i(E) = 0$ unless $-r < i \leq 0$.

The sheaf $\mathrm{H}^0(\widetilde{E})$ is g^*-invariant, so by the lemma, is isomorphic to p^*M for some sheaf M on X. There is a canonical morphism $\widetilde{E} \to \mathrm{H}^0(\widetilde{E})$, and hence a triangle

$$\widetilde{E} \longrightarrow \mathrm{H}^0(\widetilde{E}) \xrightarrow{\tilde{f}} \widetilde{F} \longrightarrow \widetilde{E}[1],$$

in $\mathrm{D}(\widetilde{X})$, where $\widetilde{F}$ has $r-1$ non-zero cohomology objects. Applying g^* we obtain an isomorphic triangle, because there is a commutative diagram

$$\begin{array}{ccc} \widetilde{E} & \longrightarrow & \mathrm{H}^0(\widetilde{E}) \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{H}^0(s)} \\ g^*\widetilde{E} & \longrightarrow & \mathrm{H}^0(g^*\widetilde{E}). \end{array}$$

It follows that $g^*\widetilde{F} \cong \widetilde{F}$, and so, by induction, $\widetilde{F} \cong p^*F$ for some object F of $\mathrm{D}(X)$.

The lemma below then implies that $\tilde{f} = p^*(f)$ for some morphism $f : M \to F$ of $\mathrm{D}(X)$. Thus there is an object E of $\mathrm{D}(X)$ and a triangle

$$E \longrightarrow M \xrightarrow{f} F \longrightarrow E[1].$$

Applying p^* one sees that $p^*E \cong \widetilde{E}$. ∎

Lemma 2.6. *Let M be an $\mathcal{O}_X$-module, and let F be an object of $\mathrm{D}(X)$. Let*

$$\tilde{f} : p^*M \to p^*F$$

be a morphism of $\mathrm{D}(\widetilde{X})$ such that $g^*(\tilde{f}) = \tilde{f}$. Then $\tilde{f} = p^*(f)$ for some morphism $f : M \to F$ of $\mathrm{D}(X)$.

Proof. Replace F by an injective resolution

$$\cdots \longrightarrow I^{-1} \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots,$$

as in [6], Lemma I.4.6. Then $\tilde{f}$ is represented by a morphism of $\mathcal{O}_{\widetilde{X}}$-modules

$$s : p^*M \to p^*I^0.$$

If V is a finite-dimensional vector space on which G acts, define operators A and B by

$$A = \sum_{i=0}^{n-1} (g^i)^*, \qquad B = 1 - g^*.$$

Then since $AB = BA = 0$, it is easy to check that $\ker A = \operatorname{im} B$.

Take V to be the image of the map

$$p^*(d^{-1})_* : \mathrm{Hom}_{\widetilde{X}}(p^*M, p^*I^{-1}) \longrightarrow \mathrm{Hom}_{\widetilde{X}}(p^*M, p^*I^0).$$

The fact that $\tilde{f}$ is G-invariant means that Bs is an element of V. Since $A(Bs) = 0$, there is an element k of V with $Bk = Bs$. Now

$$t = s - k \in \mathrm{Hom}_{\widetilde{X}}(p^*M, p^*I^0)$$

also represents $\tilde{f}$, and since $Bt = 0$, is equal to $p^*(u)$ for some $u \in \mathrm{Hom}_X(M, I^0)$. The result follows. $\square$

3. Fourier-Mukai transforms

Let X and Y be smooth projective varieties, and let $\mathcal{P}$ be an object of $\mathrm{D}(Y \times X)$. Define a functor

$$\Phi^{\mathcal{P}}_{Y \to X} : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$$

by the formula

$$\Phi^{\mathcal{P}}_{Y \to X}(-) \cong \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_Y^*(-)),$$

where $Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$ are the projection maps. Functors of this form will be called *integral functors*. It is easily checked [10], Prop. 1.3, that the composite of two integral functors is again an integral functor.

A Fourier-Mukai (FM) transform relating X and Y is an exact equivalence of categories

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X).$$

Here exact means commuting with the translation functors and taking triangles to triangles. It was proved by Orlov [14], Thm. 2.2, that for any such equivalence there is an object $\mathcal{P}$ of $\mathrm{D}(Y \times X)$, unique up to isomorphism, such that Φ is isomorphic to the functor $\Phi^{\mathcal{P}}_{Y \to X}$. We call $\mathcal{P}$ the *kernel* of the transform Φ.

We shall need the following facts concerning FM transforms.

3.1. Recall the definition of a Serre functor on a triangulated category, [2], pp. 5-6. If X is a smooth projective variety, the functor

$$S_X(-) = (\omega_X \otimes -)[\dim X],$$

is a Serre functor on $\mathrm{D}(X)$. When a Serre functor exists it is unique up to isomorphism, so any FM transform relating smooth projective varieties X and Y must commute with the functors S_X and S_Y. It follows from this that the canonical bundles of X and Y have the same order.

3.2. If $\mathcal{P}$ is the kernel of a FM transform relating X and Y, then there is an isomorphism

$$\mathcal{P} \otimes \pi_X^* \omega_X \cong \mathcal{P} \otimes \pi_Y^* \omega_Y. \tag{2}$$

Indeed, up to shifts, these objects are the kernels of the left and right adjoint functors of Φ respectively (see e.g. [4], Lemma 4.5), which, since Φ is an equivalence, must both be isomorphic to the quasi-inverse of Φ.

3.3. Suppose one has a FM transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ such that for each $y \in Y$, there is a point $f(y) \in X$ with

$$\Phi(\mathcal{O}_y) \cong \mathcal{O}_{f(y)}.$$

I claim that f defines a morphism $Y \to X$, and for some line bundle L on Y, there is an isomorphism of functors

$$\Phi(-) \cong f_*(L \otimes -).$$

To see this note that by [4], Lemma 4.3, the kernel $\mathcal{P}$ of Φ is a sheaf on $Y \times X$, flat over Y, such that for each $y \in Y$, $\mathcal{P}_y \cong \mathcal{O}_{f(y)}$. But if $\Delta \subset X \times X$ denotes the diagonal, then the sheaf $\mathcal{O}_\Delta$ is a universal sheaf parameterising structure sheaves of points of X. It follows that f is a morphism of varieties, and

$$\mathcal{P} \cong (f \times 1_X)^*(\mathcal{O}_\Delta) \otimes \pi_Y^*(L)$$

for some line bundle L on Y. The claim follows.

3.4. Many examples of FM transforms for surfaces are constructed using the following theorem, which is a simple consequence of the results of [4]. See [5] for a proof.

Theorem 3.1. *Let X be a smooth projective surface with a fixed polarisation and let Y be a 2-dimensional, complete, smooth, fine moduli space of stable, special sheaves on X. Then there is a universal sheaf $\mathcal{P}$ on $Y \times X$ and the resulting functor $\Phi = \Phi^{\mathcal{P}}_{Y \to X}$ is a FM transform.* □

We shall need the following well-known observation. Suppose we are in the situation of the theorem, and suppose that E is a stable sheaf on X with the same Chern character as the sheaves $\mathcal{P}_y$. Then I claim that E must be isomorphic to one of the $\mathcal{P}_y$. If not, for each $y \in Y$, we must have

$$\mathrm{Hom}_X(E, \mathcal{P}_y) = 0, \qquad \mathrm{Hom}_X(\mathcal{P}_y, E) = 0.$$

Since $\mathcal{P}_y$ is special, Serre duality implies that

$$\mathrm{Ext}^2_X(E, \mathcal{P}_y) = 0,$$

and since E has the same Chern character as $\mathcal{P}_y$, and Φ is an equivalence

$$\chi(E, \mathcal{P}_y) = \chi(\mathcal{P}_y, \mathcal{P}_y) = \chi(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)) = \chi(\mathcal{O}_y, \mathcal{O}_y) = 0,$$

so this is enough to show that $\mathrm{Hom}^i_X(E, \mathcal{P}_y) = 0$, for all i. This is impossible, by [4], Example 2.2, because if Ψ is the quasi-inverse of Φ,

$$\mathrm{Hom}^i_X(E, \mathcal{P}_y) = \mathrm{Hom}^i_Y(\Psi(E), \mathcal{O}_y).$$

3.5. We give a couple of well-known examples of FM transforms, which will be useful later.

Example 3.2. The first example of an FM transform for a K3 surface was the reflection functor of [11], although Mukai never explicitly mentions the fact that it is an equivalence of derived categories.

To construct it, take a K3 surface X and let $\mathcal{P}$ be the ideal sheaf $\mathcal{I}_\Delta$ of the diagonal in $X \times X$. For any $x \in X$, $\mathcal{P}_x \cong \mathcal{I}_x$. By Theorem 3.1, $\Phi^{\mathcal{P}}_{X \to X}$ is a FM transform.

Example 3.3. Let (X, ℓ) be a principally polarised Abelian surface, and let Y be the moduli space of stable sheaves on X of Chern character $(4, 2\ell, 1)$. This moduli space is fine, complete and two-dimensional, so there is a universal sheaf $\mathcal{P}$ on $Y \times X$, and the resulting functor $\Phi^{\mathcal{P}}_{Y \to X}$ is a FM transform. In fact Y is isomorphic to X. See [9], Prop. 7.1 for details.

4. Lifts of FM transforms

In this section we prove our main result, relating FM transforms on varieties with canonical bundles of finite order, to equivariant FM transforms on the canonical covers. Throughout we shall suppose that the cyclic group $G = \mathbb{Z}_n$ acts freely on two smooth projective varieties $\widetilde{X}$ and $\widetilde{Y}$, and denote the quotient morphisms by $p_X : \widetilde{X} \longrightarrow X$ and $p_Y : \widetilde{Y} \longrightarrow Y$ respectively.

Definition 4.1. A functor $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$ will be called *equivariant* if there is an automorphism $\mu : G \to G$, and an isomorphism of functors

$$g^* \circ \widetilde{\Phi} \cong \widetilde{\Phi} \circ \mu(g)^*,$$

for each $g \in G$.

Definition 4.2. Given a functor $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$, a *lift* of Φ is a functor $\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X})$ such that the following two squares of functors commute up to isomorphism

$$\begin{array}{ccc} \mathrm{D}(\widetilde{Y}) & \xrightarrow{\widetilde{\Phi}} & \mathrm{D}(\widetilde{X}) \\ {\scriptstyle p_Y^*}\uparrow\downarrow{\scriptstyle p_{Y,*}} & & {\scriptstyle p_X^*}\uparrow\downarrow{\scriptstyle p_{X,*}} \\ \mathrm{D}(Y) & \xrightarrow{\Phi} & \mathrm{D}(X), \end{array}$$

i.e. such that there are isomorphisms of functors

$$p_{X,*} \circ \widetilde{\Phi} \cong \Phi \circ p_{Y,*}, \qquad p_X^* \circ \Phi \cong \widetilde{\Phi} \circ p_Y^*. \tag{3}$$

We also say that $\widetilde{\Phi}$ *descends* to give the functor Φ.

We start with a couple of lemmas.

Lemma 4.3. *Let* $\Phi : \mathrm{D}(X) \longrightarrow \mathrm{D}(X)$ *and* $\widetilde{\Phi} : \mathrm{D}(\widetilde{X}) \longrightarrow \mathrm{D}(\widetilde{X})$ *be integral functors, such that* $\widetilde{\Phi}$ *lifts* Φ*.*

(a) *Suppose* $\Phi \cong 1_{\mathrm{D}(X)}$*. Then* $\widetilde{\Phi} \cong g_*$ *for some* $g \in G$*.*
(b) *Suppose* $\widetilde{\Phi} \cong 1_{\mathrm{D}(\tilde{X})}$*. Then* Φ *is an equivalence. If also* $p : \widetilde{X} \to X$ *is the canonical cover, then* $\Phi(-) \cong (\omega_X^{\otimes i} \otimes -)$ *for some integer* i*.*

Proof. We start with (a). Take a point $\tilde{x} \in \widetilde{X}$, and put $x = p_X(\tilde{x})$. Then $E = \widetilde{\Phi}(\mathcal{O}_{\tilde{x}})$ satisfies $p_{X,*}(E) = \mathcal{O}_x$, so $E = \mathcal{O}_{f(\tilde{x})}$ for some point $f(\tilde{x})$ in the fibre $p^{-1}(x)$. By (3.3), $f : \widetilde{X} \to \widetilde{X}$ is a morphism of varieties, and for some line bundle L on $\widetilde{X}$,

$$\widetilde{\Phi}(-) \cong f_*(L \otimes -).$$

Since $f(\tilde{x})$ always lies in the fibre $p^{-1}(x)$, $f = g$ for some $g \in G$. Now the functor $g_*^{-1} \circ \widetilde{\Phi}$ also lifts the identity, and takes $p_X^*(\mathcal{O}_X) = \mathcal{O}_{\widetilde{X}}$ to L, so in fact L must be trivial.

To prove (b), take a point $x \in X$, and a point $\tilde{x} \in \widetilde{X}$ such that $p_X(\tilde{x}) = x$. Then

$$\Phi(\mathcal{O}_x) = p_{X,*}(\mathcal{O}_{\tilde{x}}) = \mathcal{O}_x.$$

It follows from (3.3) that $\Phi \cong (L \otimes -)$ for some line bundle L on X. We must have $p_X^* L = \mathcal{O}_{\widetilde{X}}$, so if p_X is the canonical cover, the projection formula gives

$$L \otimes \Big(\bigoplus_{i=0}^{n-1} \omega_X^i\Big) = L \otimes p_{X,*}(p_X^* \mathcal{O}_X) = p_{X,*}(p_X^* L) = \bigoplus_{i=0}^{n-1} \omega_X^i,$$

hence L is a power of ω_X. ∎

Lemma 4.4. *Let* $\widetilde{\mathcal{P}}$ *and* $\mathcal{P}$ *be objects of* $\mathrm{D}(\widetilde{Y} \times \widetilde{X})$ *and* $\mathrm{D}(Y \times X)$ *respectively, such that*

$$(p_Y \times 1_X)^*(\mathcal{P}) \cong (1_{\widetilde{Y}} \times p_X)_*(\widetilde{\mathcal{P}}). \tag{4}$$

Then $\widetilde{\Phi} = \Phi^{\widetilde{\mathcal{P}}}_{\widetilde{Y} \to \widetilde{X}}$ *is a lift of* $\Phi = \Phi^{\mathcal{P}}_{Y \to X}$*.*

Proof. Put

$$f = (1_{\widetilde{Y}} \times p_X), \qquad h = (p_Y \times 1_X),$$

and consider the commutative diagram

$$\begin{array}{ccccc}
\widetilde{Y} & \xleftarrow{\pi_{\widetilde{Y}}} & \widetilde{Y}\times\widetilde{X} & \xrightarrow{\pi_{\widetilde{X}}} & \widetilde{X} \\
\| & & \Big\downarrow f & & \Big\downarrow p_X \\
\widetilde{Y} & \xleftarrow{j} & \widetilde{Y}\times X & \xrightarrow{k} & X \\
p_Y \Big\downarrow & & h \Big\downarrow & & \| \\
Y & \xleftarrow{\pi_Y} & Y\times X & \xrightarrow{\pi_X} & X.
\end{array}$$

Let E be an object of $\mathrm{D}(\widetilde{Y})$. By [6], II.5.6, II.5.12, there are natural isomorphisms

$$\begin{aligned}
p_{X,*}(\widetilde{\Phi}(E)) &= p_{X,*}\mathbf{R}\pi_{\widetilde{X},*}(\widetilde{\mathcal{P}} \overset{\mathbf{L}}{\otimes} \pi^*_{\widetilde{Y}}E) \\
&\cong \mathbf{R}k_*(f_*(\widetilde{\mathcal{P}} \overset{\mathbf{L}}{\otimes} f^*j^*E))) \cong \mathbf{R}k_*(f_*\widetilde{\mathcal{P}} \overset{\mathbf{L}}{\otimes} j^*E) \\
&\cong \mathbf{R}\pi_{X,*}(h_*(h^*\mathcal{P} \overset{\mathbf{L}}{\otimes} j^*E)) \cong \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} h_*j^*E) \\
&\cong \mathbf{R}\pi_{X,*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi^*_Y(p_{Y,*}E)) = \Phi(p_{Y,*}(E)).
\end{aligned}$$

The second isomorphism of (3) can be proved in the same way, or by taking adjoints.

□

The following theorem is the main result of this paper.

Theorem 4.5. *Let X and Y be smooth projective varieties with canonical bundles of order n, and take canonical covers*

$$p_X : \widetilde{X} \to X, \qquad p_Y : \widetilde{Y} \to Y.$$

Thus X and Y are quotients of $\widetilde{X}$ and $\widetilde{Y}$ by the cyclic group $G = \mathbb{Z}_n$. Then any FM transform

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X) \tag{5}$$

lifts to give an equivariant FM transform

$$\widetilde{\Phi} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X}). \tag{6}$$

Conversely, any equivariant FM transform (6) is the lift of some FM transform (5).

Proof. First let $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ be a FM transform, and let $\mathcal{P}$ be its kernel. Put

$$\mathcal{Q} = (p_Y \times 1_X)^*(\mathcal{P}).$$

It follows from the isomorphism (2) that $\mathcal{Q}\otimes\omega_{\widetilde{Y}\times X}\cong\mathcal{Q}$, so by Prop. 2.5, there is an object $\widetilde{\mathcal{P}}$ of $\mathrm{D}(\widetilde{Y}\times\widetilde{X})$ satisfying (4). Define $\widetilde{\Phi}=\Phi^{\widetilde{\mathcal{P}}}_{\widetilde{Y}\to\widetilde{X}}$. Then by Lemma 4.4, $\widetilde{\Phi}$ is a lift of Φ.

Let Ψ be a quasi-inverse for Φ. Then Ψ is also an FM transform and hence lifts to a functor $\widetilde{\Psi}:\mathrm{D}(\widetilde{Y})\longrightarrow\mathrm{D}(\widetilde{X})$ by the same argument. Now it is easy to check that $\widetilde{\Psi}\circ\widetilde{\Phi}$ is a lift of $\Psi\circ\Phi\cong 1_{\mathrm{D}(Y)}$. Hence, by Lemma 4.3, composing $\widetilde{\Psi}$ with g^* for some $g\in G$, we can assume that $\widetilde{\Psi}\circ\widetilde{\Phi}\cong 1_{\mathrm{D}(\widetilde{Y})}$. Similarly, $\widetilde{\Phi}\circ\widetilde{\Psi}\cong 1_{\mathrm{D}(\widetilde{X})}$, so $\widetilde{\Phi}$ is an equivalence.

Take $g\in G$ and consider the FM transform $g^*\circ\widetilde{\Phi}$. This is also a lift of Φ, so $\widetilde{\Psi}\circ g^*\circ\widetilde{\Phi}$ is a lift of $1_{\mathrm{D}(Y)}$. By Lemma 4.3 again, there is an element $\mu(g)\in H$ such that

$$g^*\circ\widetilde{\Phi}\cong\widetilde{\Phi}\circ(\mu(g))^*.$$

Clearly, the homomorphism $\mu:G\to G$ must be injective, and so by symmetry it is an isomorphism.

For the converse, let $\widetilde{\Phi}:\mathrm{D}(\widetilde{Y})\longrightarrow\mathrm{D}(\widetilde{X})$ be a FM transform with kernel $\widetilde{\Phi}$. The G-equivariance of $\widetilde{\Phi}$ is equivalent to the condition

$$(1_{\widetilde{Y}}\times g)^*(\widetilde{\mathcal{P}})\cong(\mu(g)\times 1_{\widetilde{X}})^*(\widetilde{\mathcal{P}})\quad\forall g\in G.$$

It follows that $(1_{\widetilde{Y}}\times p_X)_*(\widetilde{\mathcal{P}})$ is G-invariant so that

$$(p_Y\times 1_X)^*(\mathcal{P})\cong(1_{\widetilde{Y}}\times p_X)_*(\widetilde{\mathcal{P}})$$

for some object $\mathcal{P}$ of $\mathrm{D}(Y\times X)$. Hence by Lemma 4.4, $\widetilde{\Phi}$ lifts $\Phi=\Phi^{\mathcal{P}}_{Y\to X}$.

We must show that Φ is an equivalence of categories. Let $\widetilde{\Psi}$ be a quasi-inverse of $\widetilde{\Phi}$. Then $\widetilde{\Psi}$ is G-equivariant and hence is the lift of some integral functor $\Psi:\mathrm{D}(X)\longrightarrow\mathrm{D}(X)$. But then $\widetilde{\Psi}\circ\widetilde{\Phi}\cong 1_{\mathrm{D}(\widetilde{Y})}$ lifts $\Psi\circ\Phi$, so by Lemma 4.3, twisting Ψ by some power of ω_X, $\Psi\circ\Phi\cong 1_{\mathrm{D}(Y)}$. Similarly $\Phi\circ\Psi\cong 1_{\mathrm{D}(X)}$.

$\square$

Remark. In the situation of the theorem, it is easy to see using Lemma 4.3 that if two FM transforms $\widetilde{\Phi}_1,\widetilde{\Phi}_2$ lift a given transform Φ, then $\widetilde{\Phi}_2\cong g^*\circ\widetilde{\Phi}_1$ for some $g\in G$.

Similarly, if FM transforms Φ_1,Φ_2 both lift to give the same transform $\widetilde{\Phi}$, then $\Phi_2\cong\omega_X^i\otimes\Phi_1$ for some integer i.

A couple of points remain. Let X be a smooth projective variety X whose canonical bundle has order n, and let $p_X:\widetilde{X}\to X$ be the canonical cover. Thus X is the quotient of $\widetilde{X}$ by a free action of $G=\mathbb{Z}_n$.

Firstly, suppose there is a FM transform Φ relating X to another variety Y. Then by (3.1), ω_Y also has order n, and taking canonical covers of X and Y we are in the situation of Theorem 4.5.

Secondly, suppose there is another smooth projective variety $\widetilde{Y}$ with a free G-action, and that there is an equivariant FM transform $\widetilde{\Phi}$ relating $\widetilde{X}$ and $\widetilde{Y}$. Then I claim that the quotient morphism $p_Y : \widetilde{Y} \to Y$ is a canonical cover of $Y = \widetilde{Y}/G$, so we are again in the situation of Theorem 4.5.

To prove the claim, note that by the argument used in the proof of Theorem 4.5, the functor $\widetilde{\Phi}$ descends to give a FM transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$. By the result of (3.1) ω_Y has order n. Taking a canonical cover Y' of Y we can lift Φ to a FM transform $\Phi' : \mathrm{D}(Y') \longrightarrow \mathrm{D}(\widetilde{X})$. Now $\widetilde{\Phi}^{-1} \circ \Phi'$ is an equivariant FM transform relating Y' and $\widetilde{Y}$ which lifts the identity on $\mathrm{D}(Y)$. It follows that Y' and $\widetilde{Y}$ are isomorphic as G-spaces, hence the claim.

5. Examples

Let $\widetilde{X}$ be a smooth projective surface with a fixed polarisation and let $\widetilde{Y}$ be a complete, fine, smooth, two-dimensional moduli space of stable sheaves on $\widetilde{X}$. Then there is a universal sheaf $\widetilde{\mathcal{P}}$ on $\widetilde{Y} \times \widetilde{X}$, and by Theorem 3.1, the resulting functor

$$\widetilde{\Phi} = \Phi^{\widetilde{\mathcal{P}}}_{\widetilde{Y} \to \widetilde{X}} : \mathrm{D}(\widetilde{Y}) \longrightarrow \mathrm{D}(\widetilde{X}),$$

is a FM transform.

Assume that $\widetilde{X}$ has trivial canonical bundle (so is of either Abelian or K3 type). As we noted in (3.1), $\widetilde{Y}$ also has trivial canonical bundle. Suppose further that the cyclic group $G = \mathbb{Z}_n$ acts freely on $\widetilde{X}$ via automorphisms. Let $p_X : \widetilde{X} \to X$ denote the quotient morphism.

Applying the result of (3.3) it is easy to see that there is an algebraic action of G on the moduli space $\widetilde{Y}$ such that for each point $\tilde{y} \in \widetilde{Y}$ and each $g \in G$,

$$g^*(\widetilde{\mathcal{P}}_{\tilde{y}}) \cong \widetilde{\mathcal{P}}_{g(\tilde{y})}. \tag{7}$$

If the action of G on $\widetilde{Y}$ is free then we can form the quotient $Y = \widetilde{Y}/G$, and Theorem 4.5 shows that $\widetilde{\Phi}$ descends to give a FM transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$. The following lemma gives a purely numerical criterion for when this happens.

Lemma 5.1. *The action of G on $\widetilde{Y}$ defined above is free, if and only if the highest common factor of the integers*

$$\chi(p_X^* F, \widetilde{\mathcal{P}}_{\tilde{y}}) = \chi(p_X^* F^\vee \otimes \widetilde{\mathcal{P}}_{\tilde{y}}),$$

as F varies through all vector bundles on X is 1.

Proof. Let $E = \widetilde{\mathcal{P}}_{\tilde{y}}$. Note first that by the adjunction $p_X^* \dashv p_{X,*}$

$$\chi(p_X^* F, E) = \chi(F, p_{X,*}E).$$

If the action of G on $\widetilde{Y}$ is free then as we noted above $\widetilde{\Phi}$ descends to a transform $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$. Then if $y = p_Y(\tilde{y})$, $\Phi(\mathcal{O}_y) = p_{X,*}E$, so if Ψ is the inverse of Φ,

$$\chi(F, p_{X,*}(E)) = \chi(\Psi(F), \mathcal{O}_y).$$

Since Ψ is an equivalence the highest common factor of these integers is 1.

For the converse let g be a generator of G, and suppose that the G-action is not free, so that for some proper factor m of n, the element g^m of G fixes E. Then the sheaf

$$\bigoplus_{i=0}^{m-1} (g^i)^*(E),$$

is g^*-invariant, so by Prop. 2.5, is isomorphic to $p_X^* A$ for some sheaf A on X.

For any bundle F on X,

$$m\,\chi(p_X^* F, E) = \chi(p_X^* F, p_X^* A) = \chi(p_{X,*}p_X^*(F), A) = n\,\chi(F, A),$$

because

$$p_{X,*}p_X^*(F) \cong F \otimes (\bigoplus_{i=0}^{n-1} \omega_X^i).$$

It follows that n/m divides $\chi(p_X^* F, E)$. □

Example 5.2. Let X be an Enriques surface. Then there is a K3 surface $\widetilde{X}$ with an automorphism σ of order 2 such that X is the quotient of $\widetilde{X}$ by the 2-element group generated by σ. For any point $\tilde{x} \in \widetilde{X}$ one has

$$\sigma^*(\mathcal{I}_{\tilde{x}}) = \mathcal{I}_{\sigma(\tilde{x})},$$

so the reflection functor of Example 3.2 descends to give an FM transform

$$\Phi : \mathrm{D}(X) \longrightarrow \mathrm{D}(X),$$

This has the property that for each $x \in X$ one has an exact sequence

$$0 \longrightarrow \Phi(\mathcal{O}_x) \longrightarrow \mathcal{O}_X \oplus \omega_X \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

It is this transform which was studied in [15], §3.7.

Example 5.3. Let X be a bielliptic surface whose fundamental group is cyclic of order n. Then the canonical cover of X is a product of elliptic curves $\widetilde{X} = C_1 \times C_2$, and X is the quotient of $\widetilde{X}$ by a free action of $G = \mathbb{Z}_n$.

The original Fourier-Mukai functor of [10] never descends because the sheaf $\mathcal{O}_{\widetilde{X}} = \mathcal{F}(\mathcal{O}_0)$ is G-invariant.

Consider instead the moduli space $\widetilde{Y}$ of stable sheaves on $\widetilde{X}$ of Chern character $(4, 2\ell, 1)$, where $\ell = C_1 + C_2$ is a principal polarisation. By Lemma 5.1, the FM transform of Example 3.3 descends to give an FM transform

$$\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X),$$

such that $\Phi(\mathcal{O}_y)$ is a locally free sheaf of rank $4n$ for all $y \in Y$.

Remark. Any bielliptic surface X is the quotient of a product of elliptic curves $C_1 \times C_2$ by some finite Abelian group G, but the quotient map $C_1 \times C_2 \to X$ is only the canonical cover of X if G is cyclic. Thus in general, a FM transform $\mathrm{D}(X) \longrightarrow \mathrm{D}(X)$ will *not* lift to a transform $\mathrm{D}(C_1 \times C_2) \longrightarrow \mathrm{D}(C_1 \times C_2)$.

In [5] we shall show that if X and Y are Enriques or bielliptic surfaces, and $\Phi : \mathrm{D}(Y) \longrightarrow \mathrm{D}(X)$ is a FM transform, then X and Y are isomorphic.

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